

# QUADRATIC NONRESIDUES BELOW THE BURGESS BOUND

WILLIAM D. BANKS AND VICTOR Z. GUO

**ABSTRACT.** For any odd prime number  $p$ , let  $(\cdot|p)$  be the Legendre symbol, and let  $n_1(p) < n_2(p) < \cdots$  be the sequence of positive nonresidues modulo  $p$ , i.e.,  $(n_k|p) = -1$  for each  $k$ . In 1957, Burgess showed that the upper bound  $n_1(p) \ll_\varepsilon p^{(4\sqrt{e})^{-1}+\varepsilon}$  holds for any fixed  $\varepsilon > 0$ . In this paper, we prove that the stronger bound

$$n_k(p) \ll p^{(4\sqrt{e})^{-1}} \exp(\sqrt{e^{-1} \log p \log \log p})$$

holds for all odd primes  $p$ , where the implied constant is absolute, provided that

$$k \leq p^{(8\sqrt{e})^{-1}} \exp\left(\frac{1}{2}\sqrt{e^{-1} \log p \log \log p} - \frac{1}{2} \log \log p\right).$$

For fixed  $\varepsilon \in (0, \frac{\pi-2}{9\pi-2}]$  we also show that there is a number  $c = c(\varepsilon) > 0$  such that for all odd primes  $p$  and either choice of  $\theta \in \{\pm 1\}$ , there are  $\gg_\varepsilon y/(\log y)^\varepsilon$  natural numbers  $n \leq y$  with  $(n|p) = \theta$  provided that

$$y \geq p^{(4\sqrt{e})^{-1}} \exp(c(\log p)^{1-\varepsilon}).$$

## 1. INTRODUCTION

For any odd prime  $p$ , let  $n_1(p)$  denote the least positive quadratic nonresidue modulo  $p$ ; that is,

$$n_1(p) := \min\{n \in \mathbb{N} : (n|p) = -1\},$$

where  $(\cdot|p)$  is the *Legendre symbol*. The first nontrivial bound on  $n_1(p)$  was given by Gauss [4, Article 129], who showed that  $n_1(p) < 2\sqrt{p} + 1$  holds for every prime  $p \equiv 1 \pmod{8}$ . Vinogradov [8] proved that the bound  $n_1(p) \ll_\varepsilon p^{(2\sqrt{e})^{-1}+\varepsilon}$  holds for any  $\varepsilon > 0$ , and later, Burgess [1] extended this range by showing that the bound

$$n_1(p) \ll_\varepsilon p^{(4\sqrt{e})^{-1}+\varepsilon} \tag{1.1}$$

holds for every fixed  $\varepsilon > 0$ ; this result has not been improved since 1957.

The bound (1.1) implies that the inequality  $n_1(p) \leq p^{\frac{1}{4\sqrt{e}}+f(p)}$  holds for all odd primes  $p$  with *some* function  $f$  such that  $f(p) \rightarrow 0$  as  $p \rightarrow \infty$ . Our aim in this note is to improve the bound (1.1) and to study quadratic nonresidues that lie below  $p^{(4\sqrt{e})^{-1}+\varepsilon}$  for any fixed  $\varepsilon > 0$ . To this end, let  $n_1(p) < n_2(p) < \cdots$  be the sequence of positive nonresidues modulo  $p$ .

THEOREM 1. *The bound*

$$n_k(p) \ll p^{(4\sqrt{e})^{-1}} \exp\left(\sqrt{e^{-1} \log p \log \log p}\right) \quad (1.2)$$

*holds for all odd primes  $p$  and all positive integers*

$$k \leq p^{(8\sqrt{e})^{-1}} \exp\left(\frac{1}{2}\sqrt{e^{-1} \log p \log \log p} - \frac{1}{2} \log \log p\right),$$

*where the implied constant in (1.2) is absolute.*

In a somewhat longer range, we establish the existence of *many* nonresidues.

THEOREM 2. *Let  $\varepsilon \in (0, \xi]$  be fixed, where*

$$\xi := \frac{\pi - 2}{9\pi - 2} = 0.04344896 \dots$$

*There is a number  $c = c(\varepsilon) > 0$  such that for all odd primes  $p$  and either choice of  $\theta \in \{\pm 1\}$ , we have*

$$\#\{n \leq y : (n|p) = \theta\} \gg_\varepsilon \frac{y}{(\log y)^\varepsilon} \quad \left(y \geq p^{(4\sqrt{e})^{-1}} \exp\left(c(\log p)^{1-\varepsilon}\right)\right),$$

*where the constant implied by  $\gg_\varepsilon$  depends only on  $\varepsilon$ .*

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## 2. PRELIMINARIES

Throughout the paper, we use the symbols  $O$  and  $\ll$  with their standard meanings; any implied constants are *absolute* unless otherwise specified in the notation.

Throughout the paper, we denote

$$\lambda := \frac{5\pi - 2}{9\pi - 2} = 0.52172448 \dots, \quad \eta := \frac{1}{4} - \frac{1}{2\pi} = 0.09084505 \dots$$

The constant  $\eta$  appears in Granville and Soundararajan [5, Proposition 1], which is one of our principal tools. In view of the definition of  $\xi$  in Theorem 2, we note that the following relation holds:

$$\xi = \eta(1 - \lambda) = 2\lambda - 1. \quad (2.1)$$

In a series of papers, Burgess [2, 3] established several well known bounds on relatively short character sums of the form

$$S_\chi(M, N) := \sum_{M < n \leq M+N} \chi(n) \quad (M, N \in \mathbb{Z}, N \geq 1).$$

Here, we use a slightly stronger estimate which holds for characters of prime conductor; see Iwaniec and Kowalski [7, Equation (12.58)].

LEMMA 3. Let  $\chi$  be a primitive Dirichlet character of prime conductor  $p$ . For any integer  $r \geq 1$  we have

$$|S_\chi(M, N)| \leq 30N^{1-1/r}p^{(r+1)/4r^2}(\log p)^{1/r} \quad (M, N \in \mathbb{Z}, N \geq 1).$$

PROPOSITION 4. Let  $\chi$  be a primitive Dirichlet character of prime conductor  $p$ , and put

$$M_\chi(x) := \frac{1}{x} \sum_{n \leq x} \chi(n) \quad (x \geq 1).$$

Then, uniformly for  $c \in [0, (\log p)^{1/3}]$  we have

$$M_\chi(x) \ll (\log p)^{-c^2} \quad (x \geq p^{1/4} \exp(c\sqrt{\log p \log \log p})).$$

*Proof.* We can assume that  $c > 0$  else the result is trivial. Let  $z := e^{c\sqrt{\log p \log \log p}}$ . For any integer  $N \geq p^{1/4}z$  we have by Lemma 3:

$$|M_\chi(N)| = N^{-1}|S_\chi(0, N)| \ll N^{-1/r}p^{(r+1)/4r^2}(\log p)^{1/r} \leq p^{1/4r^2}z^{-1/r}(\log p)^{1/r}$$

for any integer  $r \geq 1$ . We choose

$$r := \left\lfloor \frac{1}{2c} \left( \frac{\log p}{\log \log p} \right)^{1/2} \right\rfloor,$$

where  $\lfloor \cdot \rfloor$  is the greatest integer function. Since  $c \leq (\log p)^{1/3}$  it follows that

$$\begin{aligned} p^{1/4r^2}z^{-1/r}(\log p)^{1/r} &= \exp \left( \frac{\log p}{4r^2} - \frac{c(\log p \log \log p)^{1/2}}{r} + \frac{\log \log p}{r} \right) \\ &= \exp \left( -c^2 \log \log p + O \left( \frac{(\log \log p)^{3/2}}{(\log p)^{1/6}} \right) \right) \ll (\log p)^{-c^2}. \end{aligned}$$

This implies the stated bound.  $\square$

### 3. PROOF OF THEOREM 2

Our proof of Theorem 2 relies on ideas of Granville and Soundararajan [5,6]. We begin with a technical lemma.

LEMMA 5. Let  $g$  be a completely multiplicative function such that  $-1 \leq g(n) \leq 1$  for all  $n \in \mathbb{N}$ . Let  $x$  be large, and suppose that  $g(p) = 1$  for all  $p \leq y := \exp((\log x)^\lambda)$ . Then, uniformly for  $1/\sqrt{e} \leq \alpha \leq 1$ , we have

$$1 - \tau(\alpha) + O((\log x)^{-\xi}) \leq M_g(x^\alpha) \leq 1 - \tau(\alpha) + \frac{1}{2}\tau(\alpha)^2 + O((\log x)^{-\xi}),$$

where

$$\tau(\alpha) := \sum_{p \leq x^\alpha} \frac{1 - g(p)}{p}.$$

*Proof.* Let  $\vartheta$  be the Chebyshev function  $\vartheta(u) := \sum_{p \leq u} \log p$ , and define

$$\mathcal{X}(t) := \frac{1}{\vartheta(y^t)} \sum_{p \leq y^t} g(p) \log p.$$

Put  $u_\alpha := (\log x^\alpha) / \log y = \alpha(\log x)^{1-\lambda}$ . Using [5, Proposition 1] and taking into account (2.1), we derive the estimate

$$M_g(x^\alpha) = \sigma(u_\alpha) + O((\log x)^{-\xi}) \quad (1/\sqrt{e} \leq \alpha \leq 1), \quad (3.1)$$

where  $\sigma$  is the unique solution to the integral equation

$$u\sigma(u) = \sigma * \mathcal{X}(u) = \int_0^u \sigma(u-t)\mathcal{X}(t) dt \quad \text{for } u > 1,$$

with the initial condition  $\sigma(u) = 1$  for  $0 \leq u \leq 1$ .

Moreover, using [5, Proposition 3.6] we see that

$$1 - I_1(u_\alpha; \mathcal{X}) \leq \sigma(u_\alpha) \leq 1 - I_1(u_\alpha; \mathcal{X}) + I_2(u_\alpha; \mathcal{X}),$$

where

$$I_1(u; \mathcal{X}) := \int_1^u \frac{1 - \mathcal{X}(t)}{t} dt,$$

$$I_2(u; \mathcal{X}) := \int_1^u \int_1^u \frac{1 - \mathcal{X}(t_1)}{t_1} \frac{1 - \mathcal{X}(t_2)}{t_2} dt_1 dt_2.$$

( $t_1 + t_2 \leq u$ )

Removing the condition  $t_1 + t_2 \leq u$  we derive that  $I_2(u; \mathcal{X}) \leq I_1(u; \mathcal{X})^2$ ; hence, in view of the trivial bound  $\tau(\alpha) \ll \log \log x$  it suffices to establish the uniform estimate

$$I_1(u_\alpha; \mathcal{X}) = \tau(\alpha) + O((\log y)^{-1}) \quad (1/\sqrt{e} \leq \alpha \leq 1). \quad (3.2)$$

For this, put  $S(v) := \sum_{p \leq v} (1 - g(p)) \log p$ , and note that

$$\begin{aligned} \tau(\alpha) &= \int_y^{x^\alpha} \frac{dS(v)}{v \log v} = \left[ \frac{S(v)}{v \log v} \right]_y^{x^\alpha} + \int_y^{x^\alpha} \frac{S(v)(\log v + 1)}{(v \log v)^2} dv \\ &= \int_y^{x^\alpha} \frac{S(v)}{v^2 \log v} dv + O((\log y)^{-1}), \end{aligned}$$

where we have used the bound  $S(v) \ll v$ . Making the change of variables  $v = y^t$ ,  $dv = y^t \log y dt$ , and taking into account that  $S(y^t) = \vartheta(y^t)(1 - \mathcal{X}(t))$ , we have

$$\tau(\alpha) = \int_1^{u_\alpha} \frac{\vartheta(y^t)}{y^t} \frac{1 - \mathcal{X}(t)}{t} dt + O((\log y)^{-1}).$$

The estimate (3.2) now follows from the Prime Number Theorem in the form  $\vartheta(y^t) = y^t + O(y^t / \log y^t)$ .  $\square$

The next statement is a variant of [6, Proposition 7.1].

PROPOSITION 6. *Let  $x$  be large, and let  $f$  be a completely multiplicative function such that  $-1 \leq f(n) \leq 1$  for all  $n \in \mathbb{N}$ . Then, uniformly for  $1/\sqrt{e} \leq \alpha \leq 1$ , we have*

$$|M_f(x^\alpha)| \leq \max \left\{ |\delta_1|, \frac{1}{2} + 2(\log \alpha)^2 \right\} + O \left( \max \left\{ |M_f(x)|, (\log x)^{-\xi} \right\} \right), \quad (3.3)$$

where

$$\delta_1 := 1 - 2 \log(1 + \sqrt{e}) + 4 \int_1^{\sqrt{e}} \frac{\log u}{u+1} du = -0.656999 \dots$$

*Proof.* We follow the proof of [6, Proposition 7.1] closely, making use of the work in [5]. Let  $y := \exp((\log x)^\lambda)$ , and let  $g$  be the completely multiplicative function defined by

$$g(p) := \begin{cases} 1 & \text{if } p \leq y, \\ f(p) & \text{if } p > y. \end{cases}$$

Using [5, Proposition 4.4] (with  $S = [-1, 1]$  and  $\varphi = \pi/2$ ) and taking into account (2.1), we derive the estimate

$$M_f(x^\alpha) = \Theta(f, y) M_g(x^\alpha) + O((\log x)^{-\xi}),$$

where

$$\Theta(f, y) := \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{f(p)}{p} \right)^{-1}.$$

Since  $|M_g(x^\alpha)| \leq 1$ , we obtain (3.3) whenever  $\Theta(f, y) \leq \frac{1}{2}$ ; thus, we can assume without loss of generality that  $\Theta(f, y) \in [\frac{1}{2}, 1]$ , and it suffices to show that

$$|M_g(x^\alpha)| \leq \max \left\{ |\delta_1|, \frac{1}{2} + 2(\log \alpha)^2 \right\} + O(B) \quad (3.4)$$

holds uniformly for  $1/\sqrt{e} \leq \alpha \leq 1$ , where

$$B := \max \left\{ |M_f(x)|, (\log x)^{-\xi} \right\}.$$

Applying Lemma 5 with  $\alpha = 1$ , we have

$$\tau(1) \geq 1 + O(B).$$

Further, by Mertens' theorem we have for  $1/\sqrt{e} \leq \alpha \leq 1$ :

$$\tau(1) - \tau(\alpha) = \sum_{x^\alpha < p \leq x} \frac{1 - g(p)}{p} \leq \sum_{x^\alpha < p \leq x} \frac{2}{p} = 2 \log \alpha + O((\log x)^{-1}).$$

Consequently,

$$\tau(\alpha) \geq 1 - 2 \log \alpha + O(B).$$

Using [5, Theorem 5.1] together with (3.1), if  $\tau(\alpha) \geq 1$  we have

$$|M_g(x^\alpha)| \leq |\delta_1| + O((\log x)^{-\xi}) = |\delta_1| + O(B).$$

On the other hand, if  $1 - 2 \log \alpha + O(B) \leq \tau(\alpha) \leq 1$  we can apply Lemma 5 again to conclude that

$$|M_g(x^\alpha)| \leq 1 - \tau(\alpha) + \frac{1}{2}\tau(\alpha)^2 + O(B) \leq \frac{1}{2} + 2(\log \alpha)^2 + O(B).$$

Putting these estimates together, we obtain (3.4), which finishes the proof.  $\square$

*Proof of Theorem 2.* Let  $\chi := (\cdot|p)$ , and let  $\varepsilon \in (0, \xi]$  and  $\theta \in \{\pm 1\}$  be fixed. Since

$$\#\{n \leq y : \chi(n) = \theta\} = \frac{1}{2}y(1 + \theta M_\chi(y) + O(p^{-1})), \quad (3.5)$$

the result is easily proved for  $y > p$  (e.g., using Proposition 4). Thus, we can assume  $y \leq p$  in what follows. Moreover, it suffices to prove the theorem for all sufficiently large primes  $p$  (depending on  $\varepsilon$ ).

Let  $\alpha \in [\frac{1}{\sqrt{e}}, 1]$  and put  $x := y^{1/\alpha}$ . Note that  $\log p \asymp \log x \asymp \log y$  since  $p^{(4\sqrt{e})^{-1}} \leq y \leq p$ . Applying Proposition 4, the bound  $M_\chi(x) \ll (\log x)^{-\varepsilon}$  holds provided that

$$x \geq p^{1/4} \exp(\varepsilon^{-1/2} \sqrt{\log p \log \log p}), \quad (3.6)$$

which we assume for the moment. Since  $\varepsilon \leq \xi$ , Proposition 6 yields the bound

$$|M_\chi(y)| = |M_\chi(x^\alpha)| \leq \max\{|\delta_1|, \frac{1}{2} + 2(\log \alpha)^2\} + c_1(\log p)^{-\varepsilon}$$

with some number  $c_1 > 0$  that depends only on  $\varepsilon$ . Taking  $\alpha := \frac{1}{\sqrt{e}} + c_1(\log p)^{-\varepsilon}$ , for all sufficiently large  $p$  (depending on  $\varepsilon$ ) we have

$$|M_\chi(y)| \leq 1 - (2\sqrt{e} - 1)c_1(\log p)^{-\varepsilon} + O_\varepsilon((\log x)^{-2\varepsilon}).$$

In particular, for some sufficiently large  $c_2 > 0$  (depending on  $\varepsilon$ ) the bound

$$|M_\chi(y)| = |M_\chi(x^\alpha)| \leq 1 - c_2(\log y)^{-\varepsilon}$$

holds. In view of (3.5) we obtain the stated result.

To verify (3.6), observe that  $\alpha^{-1} \geq \sqrt{e} - c_3(\log p)^{-\varepsilon}$  with some number  $c_3 > 0$  that depends only on  $\varepsilon$ . If  $c > 0$  and  $y \geq p^{(4\sqrt{e})^{-1}} e^{c(\log p)^{1-\varepsilon}}$ , then

$$\begin{aligned} \log x = \alpha^{-1} \log y &\geq \left(\frac{1}{4\sqrt{e}} \log p + c(\log p)^{1-\varepsilon}\right) (\sqrt{e} - c_3(\log p)^{-\varepsilon}) \\ &= \frac{1}{4} \log p + \left(c\sqrt{e} - \frac{c_3}{4\sqrt{e}} - cc_3(\log p)^{-\varepsilon}\right) (\log p)^{1-\varepsilon}. \end{aligned}$$

Hence, if  $c$  and  $p$  are large enough, depending only on  $\varepsilon$ , then

$$\log x \geq \frac{1}{4} \log p + \varepsilon^{-1/2} \sqrt{\log p \log \log p}$$

as required.  $\square$

#### 4. PROOF OF THEOREM 1

Let  $C > 0$  be a fixed (absolute) constant to be determined below. Put

$$E := p^{(4\sqrt{e})^{-1}} \exp(\sqrt{e^{-1} \log p \log \log p}) \quad \text{and} \quad B := E^{1/2} (\log p)^{1/2}.$$

Let  $N := n_1(p)$  and  $M := n_k(p)$ , where  $k$  is a positive integer such that

$$k \leq CE^{1/2}(\log p)^{-1/2}. \quad (4.1)$$

To prove the theorem we need to show that  $M \ll E$ .

*Case 1:*  $N \leq B$ . If the interval  $[1, 2k]$  contains at least  $k$  nonresidues, then

$$M \leq 2k \ll E^{1/2}(\log p)^{-1/2} \ll E$$

and we are done. If the interval  $[1, 2k]$  contains fewer than  $k$  nonresidues, then  $[1, 2k]$  contains at least  $k$  residues  $m_1, \dots, m_k$ . Therefore,  $Nm_1, \dots, Nm_k$  are all nonresidues in  $[1, 2kB]$ , and we have (using (4.1) and the definition of  $B$ )

$$M \leq 2kB \ll E.$$

*Case 2:*  $N > B$ . Applying Theorem 2 with  $\varepsilon := \xi$ ,  $y := B^{5/2}$  and  $\theta := -1$ , there is an absolute constant  $c_1 > 0$  such that

$$\#\{n \leq B^{5/2} : (n|p) = -1\} \gg \frac{B^{5/2}}{(\log B)^\xi}$$

provided that

$$B^{5/2} \geq p^{(4\sqrt{e})^{-1}} \exp(c_1(\log p)^{1-\xi}).$$

Since  $B^{5/2} > E^{5/4}$  the latter inequality is easily satisfied for all large  $p$ ; thus, if  $p$  is large enough, then the  $k$ -th nonresidue  $M = n_k(p)$  satisfies

$$N \leq M \leq B^{5/2} < N^{5/2}.$$

Let  $x \in (M, N^3)$ , and note that  $\log x \asymp \log p \asymp \log N$ . Following an idea of Vinogradov, we see that the inequality  $x < N^3$  implies that every nonresidue  $n \leq x$  can be uniquely represented in the form  $n = qm$ , where  $q$  is a prime nonresidue, and  $m$  is a positive integer residue not exceeding  $x/q$ ; this leads to the lower bound

$$\sum_{n \leq x} (n|p) \geq x - 2 \sum_{\substack{N \leq q \leq x \\ (q|p) = -1}} \frac{x}{q} + O(1).$$

Since  $M = n_k(p)$ , there are at most  $k$  prime nonresidues in  $[N, M]$ ; thus,

$$\sum_{n \leq x} (n|p) \geq x - \frac{2kx}{N} - 2 \sum_{M < q \leq x} \frac{x}{q} + O(1).$$

Recalling that  $N > B = E^{1/2}(\log p)^{1/2}$  and using (4.1) together with the Prime Number Theorem, we derive the lower bound

$$\sum_{n \leq x} (n|p) \geq x \left( 1 - \frac{2C}{\log p} - 2 \log \frac{\log x}{\log M} \right) + O\left( \frac{x}{(\log x)^{100}} \right).$$

Now let  $x := e^{-3C} M^{\sqrt{e}}$ . Since  $-2 \log(1 - t) \geq 2t$  for all  $t \in [0, \frac{1}{2}]$ , for any sufficiently large  $p$  (depending on the choice of  $C$ ) we have

$$1 - 2 \log \frac{\log x}{\log M} = -2 \log \left( 1 - \frac{3C}{\sqrt{e} \log M} \right) \geq \frac{6C}{\sqrt{e} \log M},$$

and thus

$$\frac{1}{x} \sum_{n \leq x} (n|p) \geq \frac{6C}{\sqrt{e} \log M} - \frac{2C}{\log p} + O\left(\frac{1}{(\log x)^{100}}\right).$$

Since  $M \leq B^{5/2} \leq p$  for all large  $p$ , it follows that

$$\frac{1}{x} \sum_{n \leq x} (n|p) \geq \frac{C}{\log p}$$

if  $p$  is large enough (depending on  $C$ ). On the other hand, using Proposition 4 with  $c = 1$ , we see that there is an absolute constant  $C_0 > 0$  such that

$$\frac{1}{x} \sum_{n \leq x} (n|p) \leq \frac{C_0}{\log p}$$

whenever  $x \geq p^{1/4} e^{\sqrt{\log p \log \log p}}$ . If  $C$  is initially chosen so that  $C > C_0$ , then these two bounds are incompatible unless

$$e^{-3C} M^{\sqrt{e}} < p^{1/4} \exp(\sqrt{\log p \log \log p}).$$

The theorem follows.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211 USA  
E-mail address: bankswd@missouri.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211 USA  
E-mail address: zgmbf@mail.missouri.edu